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GENERALIZED COMPOSITE HURWITZ RINGS AS ARCHIMEDEAN DOMAINS

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ABSTRACT. Let $\mathcal{D} = (D_n)_{n\geq 0}$ be an ascending chain of integral domains with characteristic zero, $\mathcal{I} = (I_n)_{n\geq 1}$ an ascending chain of nonzero proper ideals of D_0, and $\mathrm{H}(\mathcal{D})$ and $\mathrm{H}(D_0, \mathcal{I})$ (respectively, $\mathrm{h}(\mathcal{D})$ and $\mathrm{h}(D_0, \mathcal{I})$) the generalized composite Hurwitz series rings (respectively, generalized composite Hurwitz polynomial rings). In this article, we give equivalent conditions for the rings $\mathrm{H}(\mathcal{D})$, $\mathrm{h}(\mathcal{D})$, $\mathrm{H}(D_0, \mathcal{I})$ and $\mathrm{h}(D_0, \mathcal{I})$ to be Archimedean domains.

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1. INTRODUCTION

1.1. **Generalized composite Hurwitz rings.** In commutative algebra, the research of ring extensions has been of interest. Among ring extensions, the polynomial extension and the power series extension have been studied by many mathematicians and have had important applications (see [1]).

Let $\mathcal{D} = (D_n)_{n\geq 0}$ be an ascending chain of commutative rings with identity and let $\mathrm{H}(\mathcal{D}) = \{\sum_{i=0}^{\infty} a_i X^i \mid a_i \in D_i \text{ for all } i \geq 0\}$. Define the addition $+$ and the multiplication $*$ as follows: For $f = \sum_{i=0}^{\infty} a_i X^i, g = \sum_{i=0}^{\infty} b_i X^i \in \mathrm{H}(\mathcal{D})$,

$$f + g = \sum_{i=0}^{\infty}(a_i + b_i)X^i \text{ and } f * g = \sum_{i+j=n} c_n X^n,$$

where $c_n = \sum_{i+j=n} \binom{n}{i} a_i b_j$. Then we can easily see that $\mathrm{H}(\mathcal{D})$ becomes a commutative ring with identity and we call it the *generalized composite Hurwitz series ring* with respect to $\mathcal{D}$. Let $\mathrm{h}(\mathcal{D})$ be the subset of $\mathrm{H}(\mathcal{D})$ consisting of all polynomials. Then $\mathrm{h}(\mathcal{D})$ is a subring of $\mathrm{H}(\mathcal{D})$ and is called the *generalized composite Hurwitz polynomial ring* with respect to $\mathcal{D}$. If $D_n = D_1$ for all $n \geq 2$, then we write $\mathrm{H}(D_0, D_1)$ and $\mathrm{h}(D_0, D_1)$ instead of $\mathrm{H}(\mathcal{D})$ and $\mathrm{h}(\mathcal{D})$, respectively. If $D_n = D_0$ for all $n \geq 1$, then the generalized composite Hurwitz rings $\mathrm{H}(\mathcal{D})$ and $\mathrm{h}(\mathcal{D})$ are precisely the same as the Hurwitz series ring $\mathrm{H}(D_0)$ and the Hurwitz polynomial ring $\mathrm{h}(D_0)$, respectively.

The notion of Hurwitz series rings was first introduced by Keigher. In [5], Keigher introduced a variant of the ring of power series and studied some of its properties. In [6], he called such a ring the Hurwitz series ring and examined ring theoretic properties of the Hurwitz series ring.

Let D_0 be a commutative ring with identity, $\mathcal{I} = (I_n)_{n\geq 1}$ an ascending chain of nonzero proper ideals of D_0, and $\mathrm{H}(D_0, \mathcal{I}) = \{\sum_{i=0}^{\infty} a_i X^i \mid a_0 \in D_0$

and $a_m \in I_m$ for all $m \geq 1\}$. Then $\mathrm{H}(D_0, \mathcal{I})$ is a proper subring of $\mathrm{H}(D_0)$ and we call it the *generalized composite Hurwitz series ring with respect to* $\mathcal{I}$. Let $\mathrm{h}(D_0, \mathcal{I})$ be the subset of all polynomials in $\mathrm{H}(D_0, \mathcal{I})$. Then $\mathrm{h}(D_0, \mathcal{I})$ is a subring of $\mathrm{H}(D_0, \mathcal{I})$ and is called the *generalized composite Hurwitz polynomial ring with respect to* $\mathcal{I}$. Let I be a nonzero proper ideal of D_0. If $I_n = I$ for all $n \geq 1$, then we write $\mathrm{H}(D_0, I)$ and $\mathrm{h}(D_0, I)$ instead of $\mathrm{H}(D_0, \mathcal{I})$ and $\mathrm{h}(D_0, \mathcal{I})$, respectively.

Let $\mathcal{D} = (D_n)_{n\geq 0}$ be an ascending chain of commutative rings with identity and let $\mathcal{I} = (I_n)_{n\geq 1}$ be an ascending chain of nonzero proper ideals of D_0. Then it is clear that $D_0 \subsetneq \mathrm{H}(D_0, \mathcal{I}) \subsetneq \mathrm{H}(D_0) \subseteq \mathrm{H}(\mathcal{D})$ and $D_0 \subsetneq \mathrm{h}(D_0, \mathcal{I}) \subsetneq \mathrm{h}(D_0) \subseteq \mathrm{h}(\mathcal{D})$.

The readers can refer to [2] and [6] for Hurwitz series rings and to [7] and [9] for (generalized) composite Hurwitz series rings.

1.2. **Archimedean rings.** Let D be an integral domain. Recall that D is an *Archimedean domain* if for any nonzero nonunit d of D, $\bigcap_{n\geq 1} d^n D = (0)$. It is clear that an integral domain satisfying the ascending chain condition on principal ideals is an Archimedean domain [3, Remark 1.1].

In [10], the authors gave equivalent conditions for the composite Hurwitz rings $\mathrm{H}(D, E)$ and $\mathrm{h}(D, E)$ to satisfy the ascending chain condition on principal ideals, where $D \subseteq E$ is an extension of integral domains with characteristic zero. In [7], the authors generalized the result in [10] and studied when the generalized composite Hurwitz rings $\mathrm{H}(\mathcal{D})$ and $\mathrm{h}(\mathcal{D})$ satisfy the ascending chain condition on principal ideals, where $\mathcal{D} = (D_n)_{n\geq 0}$ is an ascending chain of integral domains with characteristic zero. They showed that $\mathrm{H}(\mathcal{D})$ satisfies the ascending chain condition on principal ideals if and only if $\mathrm{h}(\mathcal{D})$ satisfies the ascending chain condition on principal ideals, if and only if $\bigcap_{m\geq 1} a_1 \cdots a_m D_n = (0)$ for each $n \in \mathbb{N}$ and each infinite sequence $(a_m)_{m\geq 1}$ consisting of nonzero nonunits of D_0 [7, Theorem 2.2].

In [8], the author characterized when the composite Hurwitz series rings $\mathrm{H}(D, E)$ and $\mathrm{H}(D, I)$ and the composite Hurwitz polynomial rings $\mathrm{h}(D, E)$ and $\mathrm{h}(D, I)$ are Archimedean domains, where $D \subseteq E$ is an extension of integral domains with characteristic zero and I is a nonzero proper ideal of D. In fact, it was shown that $\mathrm{H}(D, E)$ is an Archimedean domain if and only if $\mathrm{h}(D, E)$ is an Archimedean domain, if and only if $\bigcap_{n\geq 1} d^n E = (0)$ for every nonzero nonunit d of D [8, Theorem 3]; and $\mathrm{H}(D, I)$ is an Archimedean domain if and only if $\mathrm{h}(D, I)$ is an Archimedean domain, if and only if D is an Archimedean domain [8, Theorem 6].

The purpose of this article is to study when the generalized composite Hurwitz series rings $\mathrm{H}(\mathcal{D})$ and $\mathrm{H}(D_0, \mathcal{I})$ and the generalized composite Hurwitz polynomial rings $\mathrm{h}(\mathcal{D})$ and $\mathrm{h}(D_0, \mathcal{I})$ are Archimedean domains, where $\mathcal{D} = (D_n)_{n\geq 0}$ is an ascending chain of integral domains with characteristic zero and $\mathcal{I} = (I_n)_{n\geq 1}$ is an ascending chain of nonzero proper ideals of D_0. More precisely, we show that $\mathrm{H}(\mathcal{D})$ is an Archimedean domain if and only if $\mathrm{h}(\mathcal{D})$ is an Archimedean domain, if and only if for each $n \in \mathbb{N}$ and each nonzero nonunit a in D_0, $\bigcap_{k\geq 1} a^k D_n = (0)$ (Theorem 3). We also prove that $\mathrm{H}(D_0, \mathcal{I})$ is an Archimedean domain if and only if $\mathrm{h}(D_0, \mathcal{I})$ is an Archimedean domain, if and only if D_0 is an Archimedean domain (Theorem

7). These results are generalizations of [8, Theorems 3 and 6]. Finally, we give some examples (Examples 8 and 9).

2. Main results

We start this section with a characterization of units in the generalized composite Hurwitz series ring $\mathrm{H}(\mathcal{D})$ and the generalized composite Hurwitz polynomial ring $\mathrm{h}(\mathcal{D})$.

Lemma 1. ([7, Lemma 1.5]) *Let $\mathcal{D} = (D_n)_{n\geq 0}$ be an ascending chain of commutative rings with identity. Then the following assertions hold.*

(1) *$\sum_{i=0}^{\infty} a_i X^i \in H(\mathcal{D})$ is a unit if and only if a_0 is a unit in D_0.*
(2) *$\sum_{i=0}^{n} a_i X^i \in h(\mathcal{D})$ is a unit if and only if a_0 is a unit in D_0 and for each $i \in \{1, \dots, n\}$, some power of a_i is with torsion.*

Let $\mathcal{D} = (D_n)_{n\geq 0}$ be an ascending chain of commutative rings with identity and let $f = \sum_{i=0}^{\infty} a_i X^i \in \mathrm{H}(\mathcal{D})$. Then the *order* of f is the smallest nonnegative integer m such that $a_m \neq 0$ and is denoted by $\mathrm{ord}(f)$. If $f = 0$, then we define the order of f to be ∞. If $f \in \mathrm{h}(\mathcal{D})$, then the *degree* of f is the largest nonnegative integer m such that $a_m \neq 0$ and is denoted by $\deg(f)$. If $f = 0$, then we define the degree of f to be ∞. To distinguish the case of Hurwitz rings, we denote the nth Hurwitz power of f by $f^{(n)}$.

To characterize the generalized composite Hurwitz rings as Archimedean domains, we need the following lemma.

Lemma 2. *Let $\mathcal{D} = (D_n)_{n\geq 0}$ be an ascending chain of integral domains with characteristic zero. Then the following assertions hold.*

(1) *If f is a nonzero nonunit of $H(\mathcal{D})$ which has the positive order, then $\bigcap_{k\geq 1} f^{(k)} * H(\mathcal{D}) = (0)$.*
(2) *If f is a nonzero nonunit of $h(\mathcal{D})$ which has the positive degree, then $\bigcap_{k\geq 1} f^{(k)} * h(\mathcal{D}) = (0)$.*

Proof. (1) Let f be a nonzero nonunit of positive order in $\mathrm{H}(\mathcal{D})$ and let $g \in \bigcap_{k\geq 1} f^{(k)} * \mathrm{H}(\mathcal{D})$. Then for each $k \geq 1$, there exists an element $h_k \in \mathrm{H}(\mathcal{D})$ such that $g = f^{(k)} * h_k$. Note that $\mathrm{H}(\mathcal{D})$ is an integral domain [7, Lemma 1.1]; so we obtain

$$\mathrm{ord}(g) = k \cdot \mathrm{ord}(f) + \mathrm{ord}(h_k) \geq k$$

for all $k \geq 1$. Hence $\mathrm{ord}(g) = \infty$, which indicates that $g = 0$. Thus $\bigcap_{k\geq 1} f^{(k)} * \mathrm{H}(\mathcal{D}) = (0)$.

(2) Let f be a nonzero nonunit of positive degree in $\mathrm{h}(\mathcal{D})$ and let $g \in \bigcap_{k\geq 1} f^{(k)} * \mathrm{h}(\mathcal{D})$. Then for each $k \geq 1$, we can choose an element $h_k \in \mathrm{h}(\mathcal{D})$ such that $g = f^{(k)} * h_k$. Note that $\mathrm{h}(\mathcal{D})$ is an integral domain [7, Lemma 1.1]; so we obtain

$$\deg(g) = k \cdot \deg(f) + \deg(h_k) \geq k$$

for all $k \geq 1$. Hence $\deg(g) = \infty$, which implies that $g = 0$. Thus $\bigcap_{k\geq 1} f^{(k)} * \mathrm{h}(\mathcal{D}) = (0)$. ☐

We are now ready to give equivalent conditions for the generalized composite Hurwitz rings $\mathrm{H}(\mathcal{D})$ and $\mathrm{h}(\mathcal{D})$ to be Archimedean domains.

Theorem 3. *Let $\mathcal{D} = (D_n)_{n\geq 0}$ be an ascending chain of integral domains with characteristic zero. Then the following statements are equivalent.*

(1) *$H(\mathcal{D})$ is an Archimedean domain.*
(2) *$h(\mathcal{D})$ is an Archimedean domain.*
(3) *For each $n \in \mathbb{N}$, $H(D_0, D_n)$ is an Archimedean domain.*
(4) *For each $n \in \mathbb{N}$, $h(D_0, D_n)$ is an Archimedean domain.*
(5) *For each $n \in \mathbb{N}$ and each nonzero nonunit a in D_0, $\bigcap_{k\geq 1} a^k D_n = (0)$.*

Proof. (1) ⇒ (2) Let f be a nonzero nonunit of $\mathrm{h}(\mathcal{D})$. If $f(0)$ is a nonunit of D_0, then by Lemma 1(1), f is a nonunit of $\mathrm{H}(\mathcal{D})$. Since $\mathrm{H}(\mathcal{D})$ is an Archimedean domain, we have

$$(0) \subseteq \bigcap_{k\geq 1} f^{(k)} * \mathrm{h}(\mathcal{D}) \subseteq \bigcap_{k\geq 1} f^{(k)} * \mathrm{H}(\mathcal{D}) = (0).$$

Hence $\bigcap_{k\geq 1} f^{(k)} * \mathrm{h}(\mathcal{D}) = (0)$. If $f(0)$ is a unit of D_0, then by Lemma 1(2), the degree of f is positive. Hence by Lemma 2(2), $\bigcap_{k\geq 1} f^{(k)} * \mathrm{h}(\mathcal{D}) = (0)$. Thus $\mathrm{h}(\mathcal{D})$ is an Archimedean domain.

(2) ⇒ (5) Let a be a nonzero nonunit in D_0. Then a is a nonunit in $\mathrm{h}(\mathcal{D})$ by Lemma 1(2). Fix an $n \in \mathbb{N}$ and let $d \in \bigcap_{k\geq 1} a^k D_n$. Since $\mathrm{h}(\mathcal{D})$ is an Archimedean domain, we obtain

$$dX^n \in \bigcap_{k\geq 1} a^k * \mathrm{h}(\mathcal{D}) = (0).$$

Hence $d = 0$. Thus $\bigcap_{k\geq 1} a^k D_n = (0)$.

(5) ⇒ (1) Let f be a nonzero nonunit in $\mathrm{H}(\mathcal{D})$. If the order of f is positive, then $\bigcap_{k\geq 1} f^{(k)} * \mathrm{H}(\mathcal{D}) = (0)$ by Lemma 2(1). Next, we suppose that the order of f is zero. If there exists a nonzero element $g = \sum_{i=0}^{\infty} a_i X^i \in \bigcap_{k\geq 1} f^{(k)} * \mathrm{H}(\mathcal{D})$, then for each $k \in \mathbb{N}$, we can find an element $h_k = \sum_{i=0}^{\infty} b_{ki} X^i \in \mathrm{H}(\mathcal{D})$ such that $g = f^{(k)} * h_k$. Note that $\mathrm{H}(\mathcal{D})$ is an integral domain [7, Lemma 1.1]; so we obtain

$$a_{\mathrm{ord}(g)} = f(0)^k b_{k\,\mathrm{ord}(h_k)} \in \bigcap_{k\geq 1} f(0)^k D_{\mathrm{ord}(h_k)+1}.$$

Note that by Lemma 1(1), $f(0)$ is a nonunit in D_0; so by the hypothesis, $\bigcap_{k\geq 1} f(0)^k D_{\mathrm{ord}(h_k)+1} = (0)$. Therefore $a_{\mathrm{ord}(g)} = 0$, which is a contradiction. Hence $\bigcap_{k\geq 1} f^{(k)} * \mathrm{H}(\mathcal{D}) = (0)$. Thus $\mathrm{H}(\mathcal{D})$ is an Archimedean domain.

(3) ⇔ (4) ⇔ (5) These equivalences were already shown in [8, Theorem 3]. □

We next study when the generalized composite Hurwitz rings $\mathrm{H}(D_0, \mathcal{I})$ and $\mathrm{h}(D_0, \mathcal{I})$ are Archimedean domains. To do this, we need the following three lemmas.

Lemma 4. *Let D_0 be a commutative ring with identity and $\mathcal{I} = (I_n)_{n\geq 1}$ an ascending chain of nonzero proper ideals of D_0. Then the following conditions are equivalent.*

(1) *$H(D_0, \mathcal{I})$ is an integral domain.*
(2) *$h(D_0, \mathcal{I})$ is an integral domain.*
(3) *D_0 is an integral domain with characteristic zero.*

Proof. (1) $\Rightarrow$ (2) It follows immediately from the fact that $\mathrm{h}(D_0, \mathcal{I})$ is a subring of $\mathrm{H}(D_0, \mathcal{I})$.

(2) $\Rightarrow$ (3) Suppose that $\mathrm{h}(D_0, \mathcal{I})$ is an integral domain. Since D_0 is a subring of $\mathrm{h}(D_0, \mathcal{I})$, D_0 is an integral domain. Let m be the characteristic of D_0 and suppose that $m \geq 2$. Then $ma = 0$ for all $a \in D_0$. Let r and s be nonzero elements of I_1 and I_{m-1}, respectively. Then $rX * sX^{m-1} = mrsX^m = 0$. However, this contradicts the assumption that $\mathrm{h}(D_0, \mathcal{I})$ is an integral domain. Thus the characteristic of D_0 is zero.

(3) $\Rightarrow$ (1) Let $f = \sum_{i=m}^{\infty} a_i X^i, g = \sum_{j=n}^{\infty} b_j X^j$ be elements of $\mathrm{H}(D_0, \mathcal{I})$ such that $a_m \neq 0$ and $b_n \neq 0$. Since D_0 is an integral domain with characteristic zero, $\binom{m+n}{m} a_m b_n X^{m+n} \neq 0$. Hence $f * g \neq 0$. Thus $\mathrm{H}(D_0, \mathcal{I})$ is an integral domain. □

Lemma 5. *Let D_0 be a commutative ring with identity and $\mathcal{I} = (I_n)_{n \geq 1}$ an ascending chain of nonzero proper ideals of D_0. Then the following assertions hold.*

(1) *$\sum_{i=0}^{\infty} a_i X^i \in H(D_0, \mathcal{I})$ is a unit if and only if a_0 is a unit in D_0.*
(2) *$\sum_{i=0}^{n} a_i X^i \in h(D_0, \mathcal{I})$ is a unit if and only if a_0 is a unit in D_0 and for each $i \in \{1, \ldots, n\}$, some power of a_i is with torsion.*

Proof. (1) Let $f = \sum_{i=0}^{\infty} a_i X^i \in \mathrm{H}(D_0, \mathcal{I})$.

($\Rightarrow$) Suppose that f is a unit in $\mathrm{H}(D_0, \mathcal{I})$. Then $f * g = 1$ for some $g = \sum_{i=0}^{\infty} b_i X^i \in \mathrm{H}(D_0, \mathcal{I})$. Hence $a_0 b_0 = 1$. Thus a_0 is a unit in D_0.

($\Leftarrow$) In order to show that f is a unit in $\mathrm{H}(D_0, \mathcal{I})$, we construct an element $g = \sum_{i=0}^{\infty} b_i X^i \in \mathrm{H}(D_0, \mathcal{I})$ such that $f * g = 1$. Since a_0 is a unit in D_0, $a_0 b_0 = 1$ for some $b_0 \in D_0$. Let $b_1 = -a_1 b_0^2$. Then $b_1 \in I_1$ and $a_0 b_1 + a_1 b_0 = 0$. Suppose that we have $b_j \in I_j$ for all $j = 1, \ldots, n$, and let $b_{n+1} = -b_0 \Big(\sum_{i=1}^{n+1} \binom{n+1}{i} a_i b_{n+1-i} \Big)$. Then $b_{n+1} \in I_{n+1}$. By setting $g = \sum_{i=0}^{\infty} b_i X^i \in \mathrm{H}(D_0, \mathcal{I})$, we deduce that $f * g = 1$. Thus f is a unit in $\mathrm{H}(D_0, \mathcal{I})$.

(2) Let $f = \sum_{i=0}^{n} a_i X^i \in \mathrm{h}(D_0, \mathcal{I})$.

($\Rightarrow$) Let f be a unit in $\mathrm{h}(D_0, \mathcal{I})$. Then f is a unit in $\mathrm{h}(D_0)$. Thus a_0 is a unit in D_0 and for each $i \in \{1, \ldots, n\}$, some power of a_i is with torsion [2, Theorem 3.1].

($\Leftarrow$) Suppose that for each $i \in \{1, \ldots, n\}$, some power of a_i is with torsion. Then $\sum_{i=1}^{n} a_i X^i$ is nilpotent [2, Theorem 2.4]. Thus f is a unit in $\mathrm{h}(D_0, \mathcal{I})$ [1, Section 1, Exercise 1]. □

Lemma 6. *Let D_0 be an integral domain with characteristic zero and $\mathcal{I} = (I_n)_{n \geq 1}$ an ascending chain of nonzero proper ideals of D_0. Then the following assertions hold.*

(1) *If f is a nonzero nonunit of $H(D_0, \mathcal{I})$ which has the positive order, then $\bigcap_{k \geq 1} f^{(k)} * H(D_0, \mathcal{I}) = (0)$.*
(2) *If f is a nonzero nonunit of $h(D_0, \mathcal{I})$ which has the positive degree, then $\bigcap_{k \geq 1} f^{(k)} * h(D_0, \mathcal{I}) = (0)$.*

Proof. While the proof can be done by a simple modification of that of Lemma 2, we insert it for the sake of completeness.

(1) Let f be a nonzero nonunit of $\mathrm{H}(D_0, \mathcal{I})$ which has the positive order and let $g \in \bigcap_{k \geq 1} f^{(k)} * \mathrm{H}(D_0, \mathcal{I})$. Then for each $k \geq 1$, there exists a suitable

element $h_k \in \mathrm{H}(D_0, \mathcal{I})$ such that $g = f^{(k)} * h_k$. Since $\mathrm{H}(D_0, \mathcal{I})$ is an integral domain by Lemma 4, we obtain

$$\mathrm{ord}(g) = k \cdot \mathrm{ord}(f) + \mathrm{ord}(h_k) \geq k$$

for all $k \geq 1$. Therefore $\mathrm{ord}(g) = \infty$, and hence $g = 0$. Thus $\bigcap_{k\geq 1} f^{(k)} * \mathrm{H}(D_0, \mathcal{I}) = (0)$.

(2) Let f be a nonzero nonunit of $\mathrm{h}(D_0, \mathcal{I})$ which has the positive degree and take any $g \in \bigcap_{k\geq 1} f^{(k)} * \mathrm{h}(D_0, \mathcal{I})$. Then for each $k \geq 1$, we can choose an element $h_k \in \mathrm{h}(D_0, \mathcal{I})$ such that $g = f^{(k)} * h_k$. Since $\mathrm{h}(D_0, \mathcal{I})$ is an integral domain by Lemma 4, we obtain

$$\deg(g) = k \cdot \deg(f) + \deg(h_k) \geq k$$

for all $k \geq 1$. Therefore $\deg(g) = \infty$, and hence $g = 0$. Thus $\bigcap_{k\geq 1} f^{(k)} * \mathrm{h}(D_0, \mathcal{I}) = (0)$. □

Now, we are ready to give necessary and sufficient conditions for the generalized composite Hurwitz rings $\mathrm{H}(D_0, \mathcal{I})$ and $\mathrm{h}(D_0, \mathcal{I})$ to be Archimedean domains.

Theorem 7. *Let D_0 be an integral domain with characteristic zero and $\mathcal{I} = (I_n)_{n\geq 1}$ an ascending chain of nonzero proper ideals of D_0. Then the following statements are equivalent.*

(1) *$H(D_0, \mathcal{I})$ is an Archimedean domain.*
(2) *$h(D_0, \mathcal{I})$ is an Archimedean domain.*
(3) *D_0 is an Archimedean domain.*
(4) *$H(D_0)$ is an Archimedean domain.*
(5) *$h(D_0)$ is an Archimedean domain.*

Proof. (1) ⇒ (2) Let f be a nonzero nonunit of $\mathrm{h}(D_0, \mathcal{I})$. If $f(0)$ is a nonunit of D_0, then by Lemma 5(1), f is a nonunit of $\mathrm{H}(D_0, \mathcal{I})$. Since $\mathrm{H}(D_0, \mathcal{I})$ is an Archimedean domain, we have

$$(0) \subseteq \textstyle\bigcap_{k\geq 1} f^{(k)} * \mathrm{h}(D_0, \mathcal{I}) \subseteq \bigcap_{k\geq 1} f^{(k)} * \mathrm{H}(D_0, \mathcal{I}) = (0).$$

Hence $\bigcap_{k\geq 1} f^{(k)} * \mathrm{h}(D_0, \mathcal{I}) = (0)$. If $f(0)$ is a unit in D_0, then by Lemma 5(2), the degree of f is positive. Hence by Lemma 6(2), $\bigcap_{k\geq 1} f^{(k)} * \mathrm{h}(D_0, \mathcal{I}) = (0)$. Thus $\mathrm{h}(D_0, \mathcal{I})$ is an Archimedean domain.

(2) ⇒ (3) Let d be a nonzero nonunit of D_0. Then by Lemma 5(2), d is a nonunit of $\mathrm{h}(D_0, \mathcal{I})$. Since $\mathrm{h}(D_0, \mathcal{I})$ is an Archimedean domain, we have

$$(0) \subseteq \textstyle\bigcap_{k\geq 1} d^k D_0 \subseteq \bigcap_{k\geq 1} d^k * \mathrm{h}(D_0, \mathcal{I}) = (0).$$

Hence $\bigcap_{k\geq 1} d^k D_0 = (0)$. Thus D_0 is an Archimedean domain.

(3) ⇒ (1) Let f be a nonzero nonunit of $\mathrm{H}(D_0, \mathcal{I})$. If f has the positive order, then by Lemma 6(1), $\bigcap_{k\geq 1} f^{(k)} * \mathrm{H}(D_0, \mathcal{I}) = (0)$. Suppose that the order of f is zero and let $g = \sum_{i=0}^{\infty} a_i X^i \in \bigcap_{k\geq 1} f^{(k)} * \mathrm{H}(D_0, \mathcal{I})$. Then for each $k \geq 1$, there exists an element $h_k = \sum_{i=0}^{\infty} b_{ki} X^i \in \mathrm{H}(D_0, \mathcal{I})$ such that $g = f^{(k)} * h_k$. Since $\mathrm{H}(D_0, \mathcal{I})$ is an integral domain by Lemma 4, we obtain

$$a_{\mathrm{ord}(g)} = f(0)^k b_{k\,\mathrm{ord}(h_k)} \in \textstyle\bigcap_{k\geq 1} f(0)^k D_0.$$

Note that by Lemma 5(1), $f(0)$ is a nonunit of D_0. Since D_0 is an Archimedean domain, $\bigcap_{k\geq 1} f(0)^k D_0 = (0)$. Hence $a_{\mathrm{ord}(g)} = 0$, which means that $g = 0$. Thus $\mathrm{H}(D_0, \mathcal{I})$ is an Archimedean domain.

(3) ⇔ (4) ⇔ (5) These equivalences can be obtained by applying Theorem 3 to the case when $D_n = D_0$ for all $n \in \mathbb{N}$. □

We end this article with some examples.

Example 8. Let D_0 be the ring of entire functions, $\{X_n \mid n \in \mathbb{N}\}$ a set of indeterminates over D_0 and for each $n \geq 1$, let $D_n = D_0[X_1, \ldots, X_n]$. Let $\mathcal{D} = (D_n)_{n\geq 0}$.

(1) It is obvious that $\mathcal{D}$ is an ascending chain of integral domains with characteristic zero.

(2) Note that D_0 is a completely integrally closed domain [4, Section 13, Exercise 16]; so D_0 is an Archimedean domain [4, Corollary 13.4]. Let a be a nonzero nonunit in D_0. Then $\bigcap_{k\geq 1} a^k D_0 = (0)$. Note that for each $n \geq 1$, we have

$$\bigcap_{k\geq 1} a^k D_n = \left(\bigcap_{k\geq 1} a^k D_0\right)[X_1, \ldots, X_n] = (0).$$

Hence by Theorem 3, $\mathrm{H}(\mathcal{D})$ and $\mathrm{h}(\mathcal{D})$ are Archimedean domains. Also, by Theorem 3, for each $n \geq 1$, $\mathrm{H}(D_0, D_n)$ and $\mathrm{h}(D_0, D_n)$ are Archimedean domains.

Example 9. Let D_0 be a one dimensional valuation domain with characteristic zero and $\mathcal{I} = (I_n)_{n\geq 1}$ an ascending chain of nonzero proper ideals of D_0. Then D_0 is a completely integrally closed domain [4, Theorem 17.5(3)]; so D_0 is an Archimedean domain [4, Corollary 13.4]. By Theorem 7, $\mathrm{H}(D_0, \mathcal{I})$ and $\mathrm{h}(D_0, \mathcal{I})$ are Archimedean domains. Also, by Theorem 7, $\mathrm{H}(D_0)$ and $\mathrm{h}(D_0)$ are Archimedean domains.

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